ENGINEERING PHYSICS

UNIT-3

ELECTRICAL CONDUCTION IN METALS

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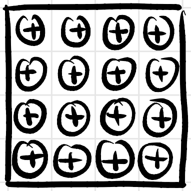
Solids

will be studing

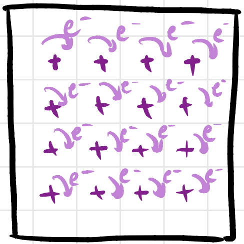
- crystalline & amorphous
- amorphous: ordering of atoms is short range
- crystalline: periodic arrangement of atoms in lattice is long-range.



solid

Crystalline Solids

Monovalent Copper



$n = 10^{28}$ atoms/m^3

+ : positive ionic core
(nucleus + tightly bounded e^-)

e^- : free electrons (sea of electrons)

- for monovalent copper, no. of free $e^- = 10^{28}$
- e^- will move with thermal velocity

$$\langle V_{Thermal} \rangle = 0 \Rightarrow \text{no current.}$$

- only drift velocity can give rise to current (emf)

Three Models to Explain Electrical Conduction

1) Classical Free Electron Theory (CFET)

- Also called Drude-Lorentz Model

2) Quantum Free Electron Theory

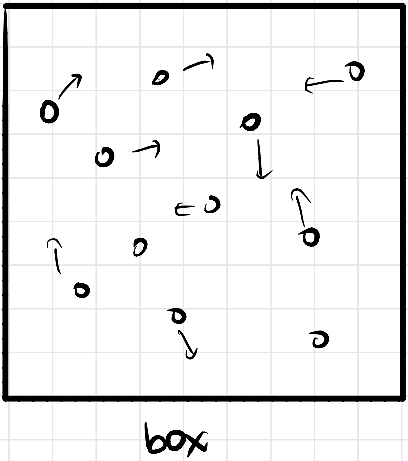
- Sommerfield Model

3) Kronig-Penney Model

CFET

Assumptions

1. Electrons in metals treated as gas molecules (free electron gas)



box

λ: mean free path

τ: relaxation time

$$\frac{3kT}{2} = \frac{1}{2} m v_{th}^2$$

$$v_{th} = \sqrt{\frac{3kT}{m}}$$

$$\langle v_{th} \rangle = 0 \Rightarrow \text{no current}$$

2. Average behaviour of electron (velocity, energy) is studied using classical statistics (MV statistics)

(other statistics: Bose-Einsten, Fermi-Dirac)

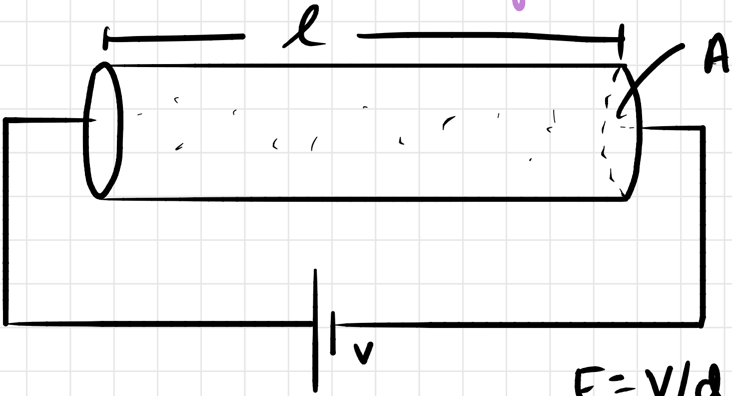
Bose-Einsten ↓ integral spin

Fermi-Dirac ↓ ½ integral spin

3. $e^- - e^-$ interactions and e^- - +ve ionic core interaction neglected.

4. e^- treated as free particles (constant potential)

Expression for Electrical conductivity



n, e
↓
no. of free e^-

$E = V/d$ Volts/m

When $\vec{E}$ is applied, the average velocity experienced by the e^- is called its drift velocity.

e^- will drift in the direction opposite to $\vec{E}$

$$I = neAv_d$$

$$v_d = \frac{eE}{m_e}\tau$$

$$\frac{I}{A} = ne\frac{eE}{m_e}\tau$$

conductivity

$$J = \frac{ne^2E}{m_e}\tau = \sigma E$$

Ohm's Law

$$\boxed{\sigma = \frac{ne^2\tau}{m_e}}$$

mean free path

$$v_m = \frac{\lambda}{\tau}$$

Demerits or Failures of CFET

1. Dependence of σ on n

Model	Experimental Results
$\sigma \propto n$ $\quad$ τ const for temp Al trivalent $\quad$ Cu monovalent conductivity of Al > Cu	conductivity of Cu > conductivity of Al

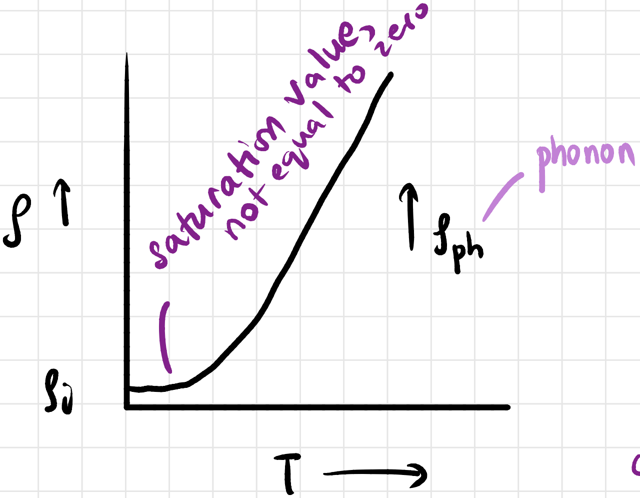
2. Dependence of σ on temperature.

Model	Experimental Results
$\sigma = \frac{ne^2}{m_e}\left(\frac{\lambda}{V_m}\right)$ $V_m = \sqrt{\frac{3kT}{m_e}}$ $\sigma = \frac{ne^2\lambda}{\sqrt{3kTm_e}}$ $\sigma \propto \frac{1}{\sqrt{T}}$	$\sigma \propto \frac{1}{T}$

3. Electronic Specific Heat

Model	Experimental Results
$U = N \frac{3}{2} kT$ $(C_v)_{ele} = \frac{dU}{dT} = N \frac{3}{2} k$	actual observed value not independent of temperature
independent of temp.	value ~100 times less than theoretical value for Al, Ag, Cu etc.
when calculated, 2 orders of magnitude higher	

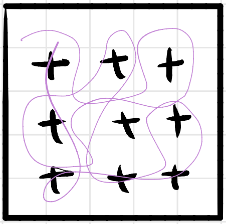
Mattheisen's Rule (Variation of resistivity with T)



$$\rho_T = \rho_i + \rho_{ph}$$

ρ_i: residual resistivity independent of temp; due to impurities/imperfections

ρ_{ph}: due to lattice vibrations depends on temp.

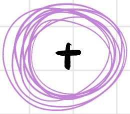


+: vibrating

zero point energy

As temp increases, area of cross-section increases

$A = \pi r^2$



r = amplitude

collisions increase, λ decreases

$$\sigma = \frac{ne^2\tau}{m_e}$$

$$\rho = \frac{m_e}{ne^2\tau}$$

$$v_{th} = \frac{\lambda}{\tau}$$

$$\rho \propto \frac{1}{\lambda} \Rightarrow T\uparrow, \rho\uparrow$$

$$\rho_T = \rho_i + \rho_{ph}$$

$$\frac{m_e}{ne\tau} = \frac{m_e}{ne\tau_i} + \frac{m_e}{ne\tau_{ph}}$$

$$\frac{1}{\tau} = \frac{1}{\tau_i} + \frac{1}{\tau_{ph}}$$

QFET

- correction to Drude model
- Given by Sommerfield

Assumptions

1. e^- are treated as quantum particles as they show quantum behaviour

Classes of Quantum Particles

1. Bosons

particles with integral spin
eg: photons

Bose particles go to highly occupied state

2. Fermions

- particles with ½ integral spin
- eg: electrons (±½)
- Fermions obey Pauli's exclusion principle

Fermi particles are identical and indistinguishable
Max. 2 particles in one energy state

Pauli's Exclusion Principle

No two electrons can occupy the same quantum state in a given energy state

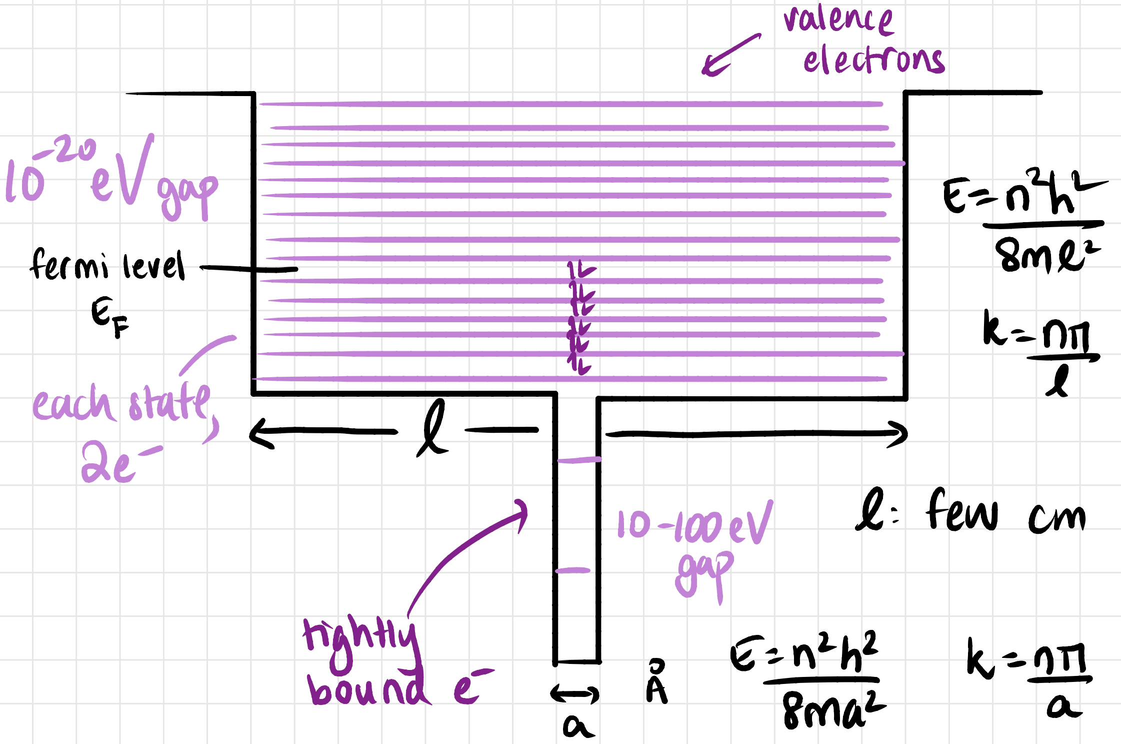
2 e^- can be accomodated in one energy state.

2. Fermions follow Fermi-Dirac statistics

$$F(E) = \frac{1}{1 + e^{\frac{E-E_F}{kT}}}$$

3. e^--e^- interactions and e^- - +ve ionic hole interactions neglected

4. e^- is a free particle moving in constant potential (quantum free electron gas)



The highest occupied energy level at $T = 0K$ is called the fermi level.

The energy of the e^- at that level is called fermi energy

At $T = 0K$, all levels above the fermi level are empty.

Probability of occupation of e^- in different energy states given by Fermi-Dirac statistics (fermi factor)

$$F(E) = \frac{1}{1 + e^{\frac{E-E_F}{kT}}}$$

fermi factor

Value of F(E) in different cases

Case 1: $E < E_F$ at $T = 0\,K$

$$F(E) = \frac{1}{1 + e^{-\infty}}$$

$$F(E) = 1$$

100% occupation below Fermi level (for $E < E_F$)

Case 2: $E > E_F$ at $T = 0\,K$

$$F(E) = \frac{1}{1 + e^{\infty}}$$

$$F(E) = 0$$

0% occupation above Fermi level (for $E > E_F$)

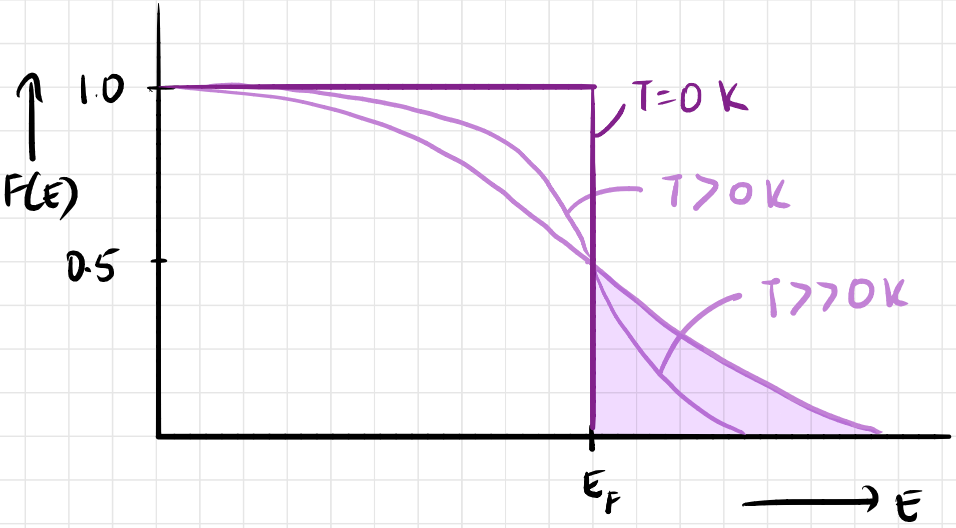
Case 3: $E = E_F$ at $T > 0\,K$

$$F(E) = \frac{1}{1 + e^{0}}$$

$$F(E) = 1/2$$

- at $T > 0\,K$, E_F is no longer the highest occupied level
- 50% occupation at E_F
- At max, 50% of e^- can occupy higher levels than E_F

Variation of Fermi Factor as a function of Temperature



From the graph, it is clear that, given thermal energy, all free e^- cannot conduct.

Only a fraction of e^- close to the fermi level can conduct

total free e^-

$$n_{eff} = N \frac{k_B T}{E_F}$$

According to quantum (Sommerfield's) theory, all free e^- are not conduction e^-

Example: E_F (Cu) = 7eV

If innermost e^- to be conduction e^-, $n_{eff} = N$

$$E_F = k_B T$$

$T_F \approx 81000$ K (Fermi Temperature)

At such high temperatures, metals are liquid (impractical)

It is the temperature at which all e^- are above E_F.

Fermi temperature is the temperature required to remove the last e^- below the Fermi level (only theoretically possible)

Q1. Find the temperatures at which the occupancy of an energy state 0.3 eV above the Fermi level has an occupancy probability of 0.01

$$F(E) = \frac{1}{1 + e^{\frac{E-E_F}{k_B T}}}$$

$$0.01 = \frac{1}{1 + e^{\frac{0.3eV}{kT}}}$$

$$100 - 1 = e^{\frac{0.3eV}{kT}}$$

$$4.595 = \frac{0.3eV}{kT}$$

$$T = 757.6 \text{ K}$$

Q2. Calculate the Fermi factor of an energy state 0.01 eV below the Fermi level at T = 500 K.

$$F(E) = \frac{1}{1 + e^{\frac{-0.01eV}{k \times 500}}} = 0.558$$

Density of States

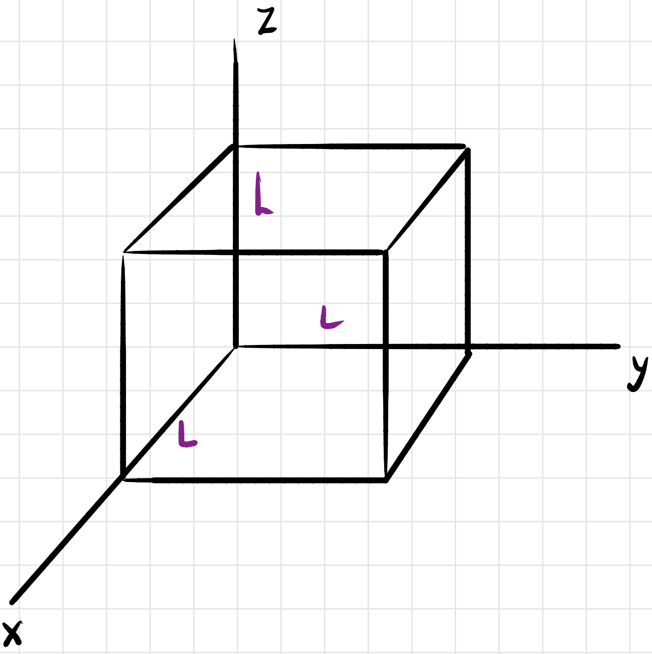
*SO Pillai Solid State

3D Solids

No. of available energy states within a given energy interval E to E+dE

$g(E)dE$ or $D(E)dE$

Consider a conducting cube (cubic crystal $a=b=c$ and $\alpha=\beta=\gamma=90°$)



- We have 10^{28} particles in a box (SWE)
- We think of the e^- as waves in a box (quantised energy)

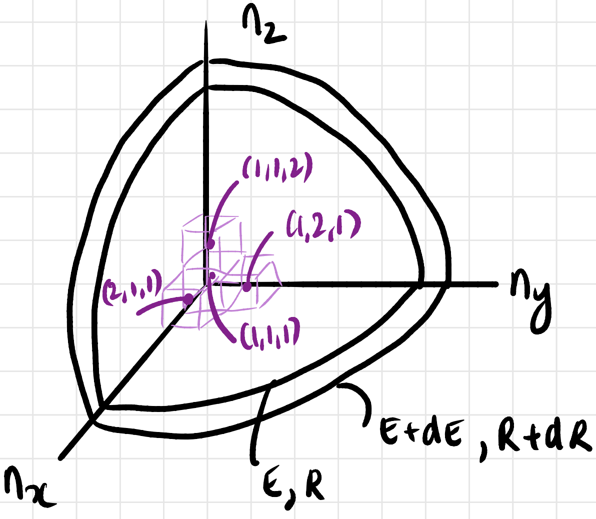
$$E_n = \frac{(n_x^2 + n_y^2 + n_z^2)h^2}{8mL^2}$$

$$\psi_n = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n_x \pi x}{L}\right)\sqrt{\frac{2}{L}}\,\sin\left(\frac{n_y \pi y}{L}\right)\sqrt{\frac{2}{L}}\,\sin\left(\frac{n_z \pi z}{L}\right)$$

(stable only when $\psi_x, \psi_y, \psi_z \neq 0$)

Phase Space

To accomodate 10^{28} electrons



We consider a radius R corresponding to all states with energy E and a sphere of radius R+dR with energy E+dE.

volume = no. of cubes = no. of points = no. of states

∴ volume = no. of states

In phase space, equation of sphere

$$R^2 = n_x^2 + n_y^2 + n_z^2$$ (three axes)

$$E = \frac{R^2 h^2}{8mL^2}$$

$$R^2 = \frac{8EmL^2}{h^2} \;;\quad R = \sqrt{8Em}\,\frac{L}{h} \;;\quad dR = \frac{L}{h}\sqrt{8m}\,\frac{1}{2\sqrt{E}}\,dE$$

No. of energy states between E and E+dE

= volume of R+dR − volume of R

$$= \frac{4\pi R^2\,dR}{8}$$

$$= \frac{\pi}{2}\left(\frac{8mL^2E}{h^2}\right)\left(\frac{L}{h}\sqrt{8m}\,\frac{1}{2\sqrt{E}}\right)dE$$

$$= \frac{\pi}{2}\left(\frac{8mL^3\sqrt{E}}{2h^3}\sqrt{8m}\right)dE$$

$$= \frac{\pi}{2}\left(\left(\frac{8mL^2}{h^2}\right)^{3/2}\frac{\sqrt{E}}{2}\right)dE$$

$$= \frac{\pi}{4}\left(\frac{8mL^2}{h^2}\right)^{3/2}\sqrt{E}\,dE$$

Each state actually can accomodate 2 distinct electron states due to spin quantum number (Pauli Exclusion Principle)

$$= 2 \times \frac{\pi}{4}\left(\frac{8mL^2}{h^2}\right)^{3/2}\sqrt{E}\,dE$$

$$= \frac{\pi}{2}\left(\frac{8mL^2}{h^2}\right)^{3/2}\sqrt{E}\,dE$$

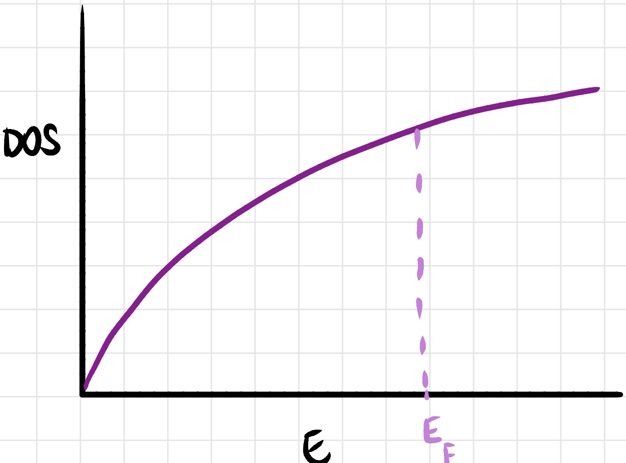
Density of states per unit volume

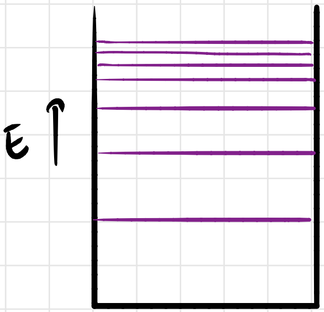
$$g(E)\,dE = \frac{\pi}{2}\left(\frac{8m}{h^2}\right)^{3/2}\sqrt{E}\,dE$$

density of states near an energy value

As $E\uparrow$, $g(E)\uparrow$

Variation of Density of states as a function of E





No of energy states upto an energy E

$$N = \int_0^E g(E)\,dE$$

$$N = \int_0^E \frac{\pi}{2}\left(\frac{8m}{h^2}\right)^{3/2} \sqrt{E}\,dE$$

$$= \frac{\pi}{2}\left(\frac{8m}{h^2}\right)^{3/2} \left[\frac{2}{3}E^{3/2}\right]_0^E$$

$$N = \frac{\pi}{3}\left(\frac{8m}{h^2}\right)^{3/2} E^{3/2}$$

No of energy states upto fermi level

$$N_F = \int_0^{E_F} g(E)\,dE$$

$$N_F = \frac{\pi}{3}\left(\frac{8m}{h^2}\right)^{3/2} (E_F)^{3/2} \longleftarrow \text{per unit volume}$$

At $T=0$, all energy states below E_F are occupied.

$\therefore$ no. of e^- present = energy states below E_F

$n \rightarrow$ electron density — electrons per unit volume

$\therefore n = N_F$

Electron Density

$$n = \frac{\pi}{3}\left(\frac{8m}{h^2}\right)^{3/2} E_F^{3/2}$$

Q3. If E_F of Cu = 7eV, find e^- concentration of Cu.

$$n = \frac{\pi}{3}\left(\frac{8m}{h^2}\right)^{3/2} (7e)^{3/2}$$

$$n = 8.411 \times 10^{28} \text{ m}^{-3}$$

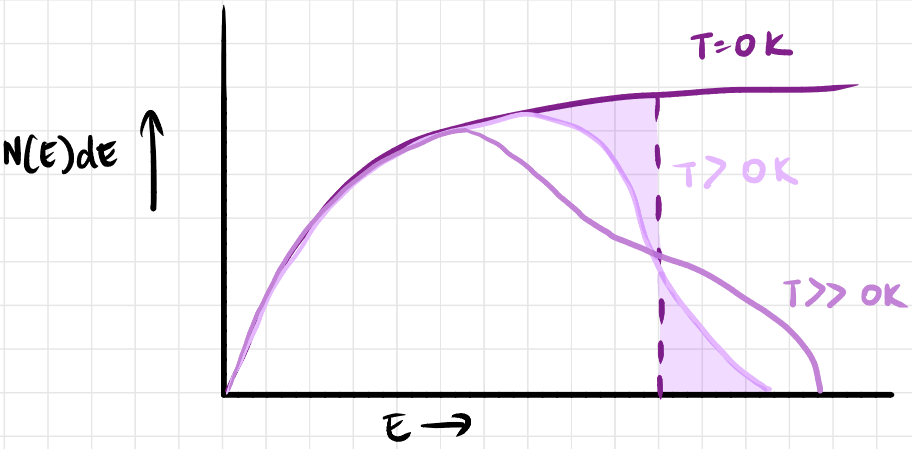
Q4. $n_{Ag} = 5.211 \times 10^{28}$ m^{-3}. $E_F = ?$

$$n = \frac{\pi}{3}\left(\frac{8m}{h^2}\right)^{3/2} (E_F)^{3/2}$$

$$\left(\frac{3n}{\pi}\right)^{2/3} \frac{h^2}{8m} = E_F$$

$$E_F = 5.09 \text{ eV}$$

Variation of Number Density as a function of Temperature



Average Energy of Electron at T= 0K

$$\text{average energy} = \frac{\int \text{no of electrons} \times \text{energy}}{\text{total no. of electrons}}$$

$$\langle E \rangle = \frac{\int_0^{E_F} g(E)\,dE\;E}{\int_0^{E_F} g(E)\,dE} = \frac{\int_0^{E_F} \left(\frac{\pi}{2}\right)\left(\frac{8m}{h^2}\right)^{3/2} \sqrt{E}\;E\;dE}{\int_0^{E_F} \left(\frac{\pi}{2}\right)\left(\frac{8m}{h^2}\right)^{3/2} \sqrt{E}\;dE}$$

$$\langle E \rangle = \left[\frac{3}{5}\frac{E^{5/2}}{E^{3/2}}\right]_0^{E_F} = \frac{3}{5}E_F$$

Qs. Find the average energy of Ag e^- ($E_F = 5.5eV$)

$$\langle E \rangle = \frac{3}{5} E_F = 3.3 eV$$

Merits of QFET

1. Dependence of σ on n

$$\sigma = \frac{ne^2 \tau}{m} \qquad v_F = \frac{\lambda_F}{\tau}$$

$$\sigma = \frac{ne^2}{m}\left(\frac{\lambda_F}{v_F}\right) \qquad v_F = \sqrt{\frac{2E_F}{m}}$$

E_F varies in metals

$$n_{Al} > n_{cu}$$

$$n_{Al} \approx 2.34\, n_{cu}$$

$$\left(\frac{\lambda_F}{V_F}\right)_{cu} \approx 3.14 \left(\frac{\lambda_F}{V_F}\right)_{Al}$$

$$\therefore \sigma_{cu} > \sigma_{Al}$$

In accordance with experiment.

2. Electronic specific heat

$$(C_v)_{ele} = \frac{du}{dT}$$

$$u = n_{eff} \frac{3}{2} k_B T$$

$$= N \frac{k_B T}{E_F} \frac{3}{2} k_B T$$

$$u = \frac{3}{2} N \frac{k_B^2}{E_F} T^2$$

$$(C_v)_{ele} = \frac{du}{dT} = 3 N \frac{k_B^2}{E_F} T$$

not independent of temperature

$(C_v)_{ele}$ for Cu is 12666 $J kg^{-1} K^{-1}$ which is in accordance with experimental results.

Q6. $(E_F)_{Cu} = 7eV$, $n = 8.28 \times 10^{28}\ m^{-3}$, find $(C_v)_{ele}$ at $T = 300\ K$.

$$(C_v)_{ele} = \frac{3 \times 8.28 \times 10^{28} \times k_B^2}{7 \times e} \times 300$$

$$= 12665\ J\ kg^{-1} K^{-1}$$

3. Dependence of σ on T

- With increase in temperature, the area of cross-section of each of the positive ionic core increases, which means the amplitude of vibrations increases
- This gives rise to more electronic collisions.
- Therefore, mean free path reduces.

$$\lambda_F \propto \frac{1}{T}$$

$$\sigma \propto \frac{1}{T}$$

$$\rho \propto \frac{1}{T}$$

Relationship between Thermal and Electrical conductivity

Weidemann-Franz Law

K : thermal conductivity
σ : electrical conductivity

$$K = \frac{1}{3} \frac{(C_v)_{ele}}{V} v_F \lambda$$

volume

$$\sigma = \frac{ne^2\tau}{m}$$

Modified Electronic Specific Heat Equation

$$(C_v)_{ele} = \frac{\pi^2}{2} N \frac{k_B^2 T}{E_F}$$

Plugging in values

$$K = \frac{1}{3} \frac{\left(\frac{\pi^2}{2} N \frac{k_B^2 T}{E_F}\right) v_F \lambda_F}{V}$$

$$= \frac{1}{3} \frac{\pi^2}{2} \frac{n k_B^2 T v_F \lambda_F}{E_F}$$

$$= \frac{1}{3} \frac{\pi^2}{2} \frac{n k_B^2 T v_F^2}{E_F} \tau$$

$$= \frac{1}{3} \frac{\pi^2}{\cancel{2}} n k_B^2 \frac{T}{\cancel{E_F}} \cdot \frac{\cancel{2} \cancel{E_F}}{m} \tau$$

$$K = \frac{1}{3} \pi^2 n k_B^2 T \frac{\tau}{m}$$

$$\sigma = \frac{n e^2 \tau}{m}$$

Relationship

$$\frac{K}{\sigma} = \frac{\pi^2}{3e^2} k_B^2 T$$ = constant at T

For any metal at a given temperature, the value of $\frac{k}{\sigma}$ is constant

$$\frac{k}{\sigma} = \boxed{\frac{\pi^2 k_B^2}{3e^2}} T$$

Lorentz number

$$\frac{k}{\sigma} = LT$$

homework:
physics assign

$$L = 2.443 \times 10^{-8} \ \mathrm{W \Omega K^{-2}}$$

Demerits of QFET

1. Generally, metals exhibit -ve Hall coefficient, but certain metals like Zn, Cd exhibit +ve Hall coefficient.

   This model could not account for this.

   Hall effect - later

2. This model could not differentiate solids as conductors, semiconductors and insulators

KRONIG-PENNEY MODEL

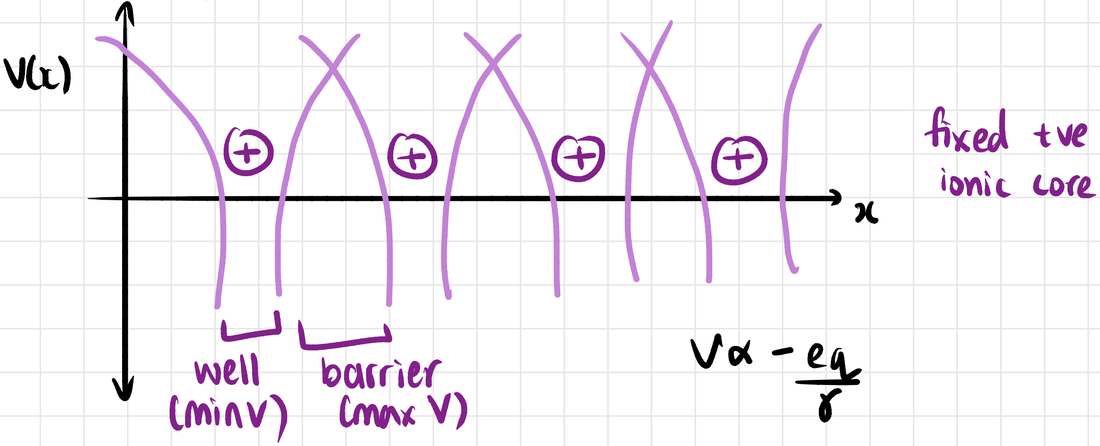
Assumption

1. Electrons are not free particles; they are partially bounded

Potential $\neq 0$

e^- inside the metal is not free; it experiences a periodic potential (series of finite wells and barriers) due to the periodicity of lattice.

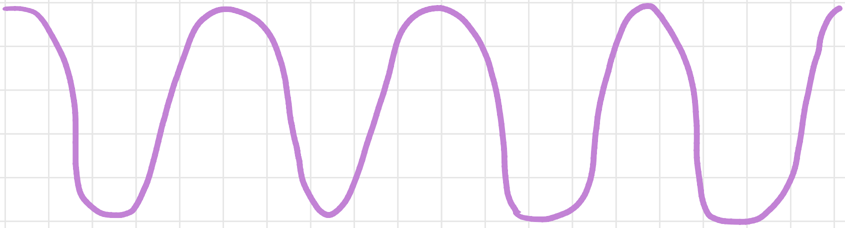
1D Crystal Lattice (Periodic Lattice)



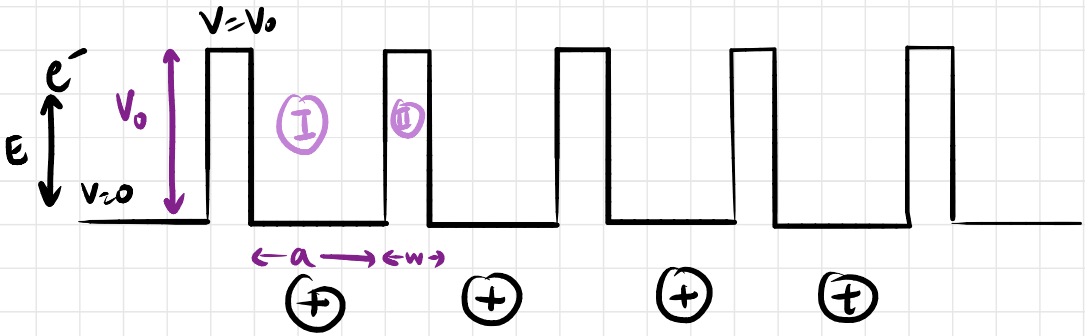
$$V \propto -\frac{eq}{r}$$

$$V(x) = V(x+a)$$

Simplifying



Simplifying further



Bloch Function

Whenever a particle experiences a periodic potential, the associated wavefunction is given by Bloch function

$$\psi(x) = V_k(x)\, e^{ikx}$$

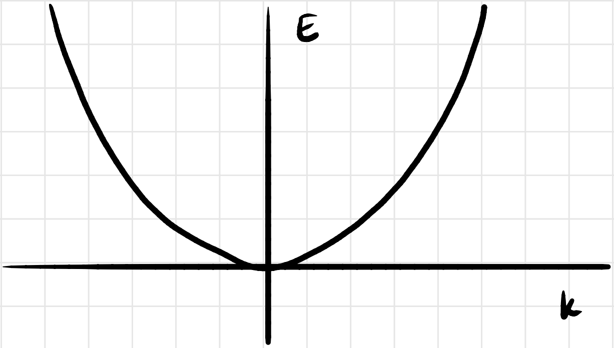
$$k = \frac{2\pi}{\lambda}$$

$$\psi(x+a) = V_k(x+a)\, e^{ik(x+a)}$$

$V_k(x)$ is a periodic function with periodicity a

$$V_k(x+a) = V_k(x)$$

For a free particle, E vs K



E can take
any value

continuous

For a partly free particle, E vs k

Solving SWE

$$\frac{d^2\psi_I}{dx^2} + \frac{2mE}{\hbar^2}\psi_I = 0 \tag{1}$$

$$\frac{d^2\psi_I}{dx^2} + \alpha^2\psi_I = 0 \qquad \text{where } \alpha^2 = \frac{2mE}{\hbar^2}$$

$$\frac{d^2\psi_{II}}{dx^2} + \frac{2m(E-V_0)}{\hbar^2}\psi_{II} = 0 \tag{2}$$

$$\frac{d^2\psi_{II}}{dx^2} + (i\beta)^2\psi_{II} = 0 \qquad \text{where } \beta^2 = \frac{2m(V_0-E)}{\hbar^2}$$

$$\psi_I = Ae^{i\alpha x} + Be^{-i\alpha x}$$

$$\psi_{II} = Ce^{\beta x} + De^{-\beta x}$$

However, these ψ_I, ψ_{II} do not give periodicity.

Therefore, we plug in the Bloch function into the differential equations (1) and (2)

$$\psi(x) = V_k e^{ikx}$$

Differentiating $\psi(x)$ and using the values in equation (1) and (2) we get two equations

Applying boundary conditions, we get four equations

can also be solved by linear algebra.

Creating Eigenvalue Equations

plane wave

$$\psi(x) = V_k(x) e^{ikx} \longrightarrow \text{eigen function}$$

$$V_k(x) = V_k(x+a) \longrightarrow \text{periodic}$$

Take a translational operator $\hat{T}$ such that

$$\hat{T}\psi(x) = \psi(x+a) \longrightarrow \text{periodicity}$$

$$\hat{T}\psi(x) = V_k(x+a) e^{ik(x+a)}$$

$$= V_k(x) e^{ikx} e^{ika}$$

$$\hat{T}\psi(x) = e^{ika}\psi(x) \longrightarrow \text{eigenvalue equation}$$

The final solution

$$P\frac{\sin\alpha a}{\alpha a} + \cos\alpha a = \cos ka \qquad \text{——} \qquad (3)$$

$$\alpha^2 = \frac{2mE}{\hbar^2}$$

$$P = \frac{mV_0 w a}{\hbar^2}$$

$V_0 w$ → barrier strength

P → scattering power

Solving (3) we can get a relationship between E and k (eigenvalues)

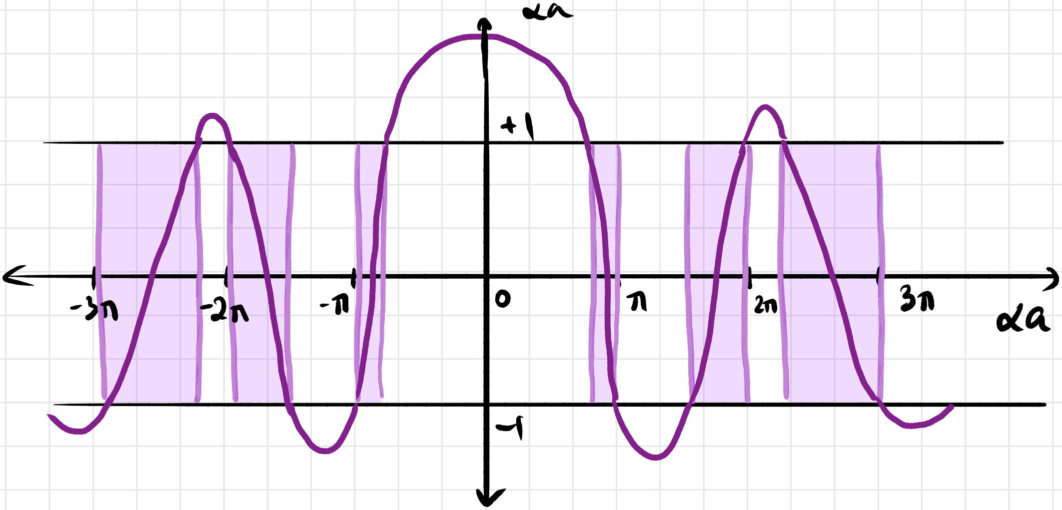
homework : desmos

Conclusions of the Model

In order to relate E and k, we find value of LHS. The values of RHS are bounded b/w 0 and 1

Plot LHS $\left(P\frac{\sin\alpha a}{\alpha a} + \cos\alpha a \right)$ wrt αa $(\alpha \to E)$

$$P\frac{\sin\alpha a}{\alpha a} + \cos\alpha a = \cos ka$$



From the graph, we notice that discontinuity arises when the y value is ±1

$$P\frac{\sin\alpha a}{\alpha a} + \cos\alpha a = \pm 1$$

$$\cos ka = \pm 1$$

$$ka = n\pi$$

$$k = \frac{n\pi}{a}$$

Whenever $k = \frac{n\pi}{a}$, gap in energies appears.

Note: k is continuous; no condition for discontinuity
α is discontinuous due to condition

α- k Correspondence

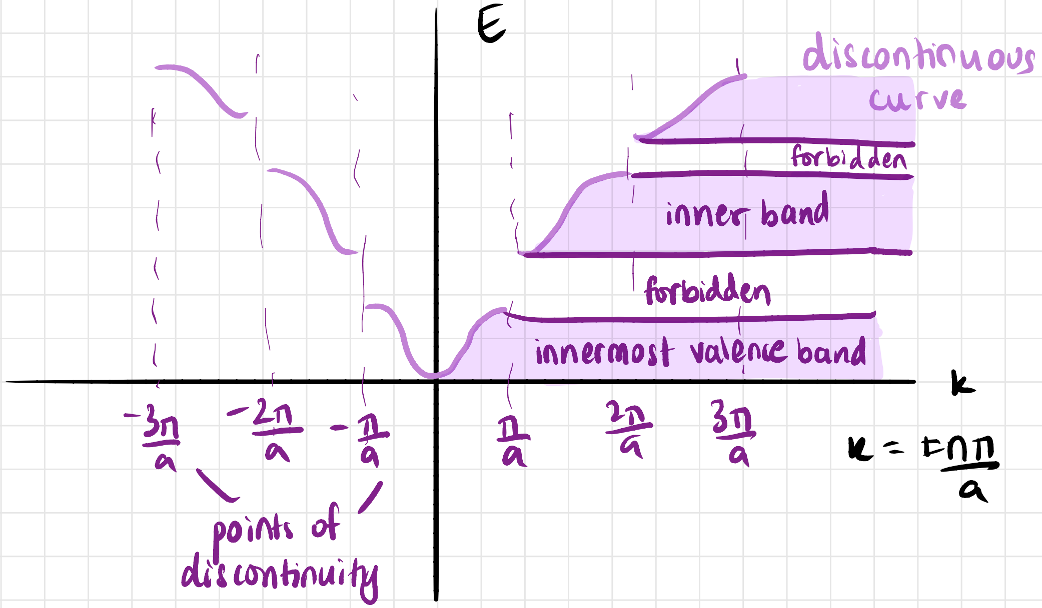
$$\alpha^2 = \frac{2mE}{\hbar^2} \quad , \quad E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

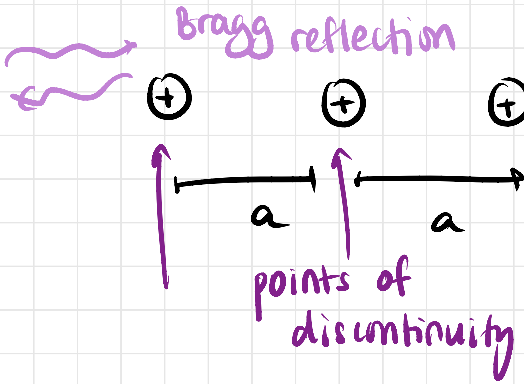
$$\alpha^2 = \frac{2m}{\hbar^2}\,\frac{\hbar^2 k^2}{2m} \Rightarrow \alpha^2 = k^2$$

α & k are physically the same and both represent momentum.

k is called crystal momentum.

1. As the value of αa increases, the width of the allowed band increases and forbidden band decreases.





2. Whenever an e^- moves in a solid, it is not free and it undergoes Bragg reflection when it encounters +ve ionic cores

ψ becomes discontinuous (particle doesn't exist there)

3. There are inner & outer bands; only outermost bands conduct.

4. In case of a metal/conductor, the VB and CB overlap (outermost band) ⇒ no energy gap
(Fermi energy in CB)

5. In case of semiconductors, the outermost band, there is a gap of ~3eV between VB and CB
(Fermi energy in band gap)

6. For insulators, there is a gap of > 5eV between VB and CB (outermost gap)
(Fermi energy in band gap)

Other Conclusions

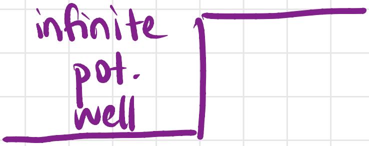
Extreme cases

Case I

$$P\frac{\sin\alpha a}{\alpha a} + \cos\alpha a = \cos ka$$

$V_0 \omega \longrightarrow \infty$ (barrier of infinite thickness)

$P \longrightarrow \infty$



to make finite, $\frac{\sin\alpha a}{\alpha a} \longrightarrow 0$

$$\alpha a = \pm n\pi$$

$$\alpha = \frac{\pm n\pi}{a}$$

But $\alpha^2 = \frac{2mE}{\hbar^2}$

$$\frac{n^2\cancel{\pi^2}}{a^2} = \frac{2mE\,4\cancel{\pi^2}}{h^2}$$

$E = \frac{n^2h^2}{8ma^2}$ infinite potential well

Energy quantised (bands)

Case II

$$V_0 w \longrightarrow 0$$
$$P \longrightarrow 0$$

$$\cos \alpha a = \cos ka$$
$$\alpha a = ka$$

$$\alpha = k$$

$$\frac{2mE}{\hbar^2} = k^2$$

$E = \frac{k^2\hbar^2}{2m}$ free particle

Energy continuous

Effective mass

For free particles,

true momentum

$$\hbar k = p = mv$$

group velocity

$$E = \frac{\hbar^2 k^2}{2m}$$

For particles in solid (crystal)

group velocity does not change

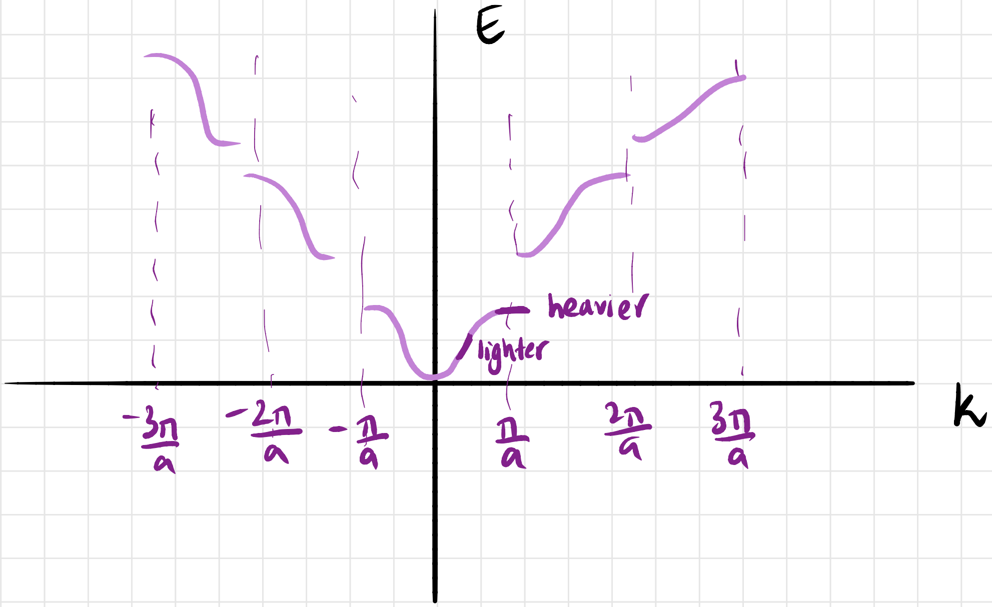
$$\frac{d^2E}{dk} = \frac{\hbar^2}{m^*}$$

$$m^* = \frac{\hbar^2}{(d^2E/dk^2)} \qquad \sigma = \frac{ne^2\tau}{m^*}$$

curvature

The effective mass depends on the curvature of E vs k graph.

Higher is the curvature, lesser will be its effective mass and vice versa



The concept of effective mass is used to study mobility of charge carriers, particularly in semiconductors

Effective mass of e^- is +ve in CB and -ve in VB.

This allows us to view conduction in VB of SC as happening due to +ve charge carriers, or holes

Effective masses of e^- in different SC

1) Ge

$$m_e^* = 0.55\, m_0$$

$$m_h^* = -0.37\, m_0$$